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The Analytic Methods of Operations Research

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The analytic methods of operations research

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Originally, the techniques used by operational research teams were borrowed from other scientific fields. However, the scope of problems addressed by this new discipline soon led to the development of special analytic methods, including such now familiar terms as linear programming, game theory, dynamic programming, queueing theory, and so forth.

This growth was accelerated by the parallel development of the high-speed digital computer and the modern concepts of solution algorithms and simulation models. When computational storage and speed were limited, emphasis was on the exploitation of a problem's special structure; as capabilities have expanded and costs have diminished, emphasis has shifted to the problems of data management for larger-scale problems. Efficient data-structure methods have led to new methods for 'unsolvable' combinatorial problems.

Concurrently, there has been an explosion in the theoretical literature. Specialty journals flourish in the various methodological areas, and conference offerings have grown to an unwieldy size. Most major universities now offer courses and degree programmes in O.R., based on topics which were unknown twenty years ago, and which use a wide variety of available textbooks.

Furthermore, it is increasingly difficult to draw a firm line between O.R. and other disciplines, as successful techniques are routinely taught and used in their fields of application, including a variety of new disciplines such as transportation and urban planning, waste management, energy analysis, environmental engineering, health care systems, etc. O.R. methods have also had a large influence on theoretical fields, such as mathematics, statistics and economics.

Recent developments in selected methodological areas are surveyed to indicate the variety and sophistication of O.R. techniques, and current research trends. Selected bibliographic references provide an introduction to the techniques, or to important new developments.

In conclusion, the current crises which have appeared within the profession are discussed, and the prospects of this now-mature field are analysed.

1. Introduction

When I received the invitation to survey the analytical tools of operational research for this meeting, I was extremely pleased – first of all, for the kind thought that I was qualified to do so; and, secondly, for the opportunity to visit again the country where operational research began over thirty years ago. However, as I considered the explosive growth of the field since that time, I became apprehensive at the thought of covering so diverse a topic in a few words.

Perhaps some of you remember the actual situation in the middle of the 1950s – the time at which most methodology began to be developed. Your Operational Research Society was about ten years old, the Operations Research Society of America was about five, and the Institute of Management Science had just been organized. The Rand Corporation had just published a book entitled *A million random digits with* 100000 *normal deviates* for use in Monte Carlo simulation; an electro-mechanical device called Queuiac was promoted in the *O.R.S.A. Journal* for emulating queueing problems. The transportation and travelling-salesman problems had just been described, and neologisms like 'sub-optimization' and 'cost-effectiveness analysis' were appearing. At M.I.T., where I was pursuing graduate studies in engineering, there was no formal curriculum in operations research; our only texts were brief notes by P. M. Morse, G. Kimball, B. O. Koopman, G. P. Wadsworth and others, Rand reports, *The theory of games and economic behavior*, by J. von Neumann & O. Morgenstern, and an esoteric paperback by Charnes, Cooper & Henderson on the optimal mixing of peanuts and cashews. Our computer used vacuum tubes, and one walked inside it for repairs; we solved what were considered very large transportation problems (60 plants, 300 customers) in about thirty minutes. Capabilities improved with the arrival of the first commercial computer, but another student's simulation of vehicular tunnel traffic took longer than real time! In 1955, I attended my first meeting of O.R.S.A. at Columbia University. There were two theoretical papers by R. E. Bellman and by J. M. Danskin, eight application papers on production scheduling and urban services, plus 24 contributed papers – I believe it was the first time that two parallel sessions were required to cover all the papers in one day. Even by 1958, a comprehensive bibliography on operations research contained only 3000 entries (Case 1958).

In contrast, the next O.R.S.A./T.I.M.S. meeting in Miami in November 1976 will, over three days, have 160 sessions with about 980 papers! There are probably over 400 texts in the methodologies of O.R. now in print, and a continually increasing number of specialty journals. A good bibliography on any of the subfields of O.R. can easily include several thousand entries. Over 40 colleges and universities in the United States now offer some form of O.R. education; over 20 of these have named departments. The technical capabilities of computers seem boundless, but our propensity to enlarge the boundaries and the scale of the models keeps pace, as we tackle national, international, and even global problems. In every dimension, the field seems limitless.

My plan of attack to reduce the survey to manageable size is as follows: First of all, I will describe briefly the major methodological areas, and present what I consider to be important recent trends, including references which seem representative and interesting, provide convenient summaries, or which might serve as gateways for further reading. I am grateful to my many colleagues who have helped to organize this bibliography; however, the final selection is mine, and no claim to completeness or showing historical priority is made.

Some methodological areas, such as control theory, are already too large to survey; other areas were excluded because they seem to have reached deadends (game theory, information theory) or because their scientific basis is still being developed (simulation, management information systems, urban and public systems). Except for a brief section on business models, applications of methodologies could, unfortunately, not be included in this limited space.

Finally, I would like to conclude by describing the influence of O.R. methodologies on other disciplines, considering some of the crises facing the profession, and giving my perspective on the prospects for this now-mature field.

We begin with a discussion of optimization methods.

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2. Unconstrained optimization

Unconstrained maximization (minimization) is the problem of finding a value x^* of a vector $x = [x_1, x_2, \ldots, x_n]$ such that $f(x^*) \geqslant f(x)$ (or $f(x^*) \leqslant f(x)$) for all x, where f is a given function, usually analytic. Most optimization methods require that f have certain smoothness and shape properties, such as continuity and concavity (convexity), so that if an x^* satisfies the necessary conditions for a local optimum, $\nabla f = [\partial f(x^*)/\partial x_i (i = 1, 2, \ldots, n)] \equiv 0$ plus 2nd-order conditions, then x^* is also globally optimal. This enables one to use local exploration to lead to the global optimum.

Assuming that the gradient ∇f is easy to calculate, the most popular algorithms are based on the idea of steep ascent (descent) – that is, from the current solution, x^0, find a new solution by moving with (against) the gradient, $x^1 = x^0 \pm S\nabla f$, where S is a positive definite matrix, varying from step to step, which may have only diagonal terms (representing current step size), or may be more general, attempting to avoid the slow convergence often encountered near x.* These methods are by now quite efficient for problems with several hundred variables. If derivatives cannot be easily computed, then a variety of direct search methods are available. Theoretical details are in Luenberger (1973) and Avriel (1976); numerical comparisons may be found in Himmelblau (1972); Powell (1970, 1971) has convenient surveys; Wilde (1964) details one-dimensional search procedures.

If equality constraints are present, then one can, in principle, use Lagrange multipliers to convert the problem to an unconstrained optimum, and apply gradient methods (Kwakernaak & Strijbos 1972; Avriel 1976). Alternatively, one can use nonlinear programming methods, described below.

Unconstrained optimization methods are useful primarily in the simple pedagogical models, or in engineering design problems. Their main value is as a foundation for more complicated methods where equality or inequality constraints are present.

3. Linear programming

Constrained linear optimization is, by any measure, one of the most successful 'new' methodologies of operational research. The problem is to optimize a linear function subject to linear inequality and equality constraints; in the usual canonical form:

$$\left.\begin{aligned} \max_x \,(\min)\; & c^T x, \\ Ax &= b, \\ x &\geqslant 0, \end{aligned}\right\} \tag{3.1}$$

where x and c are n-vectors, b is an m-vector, and A is an $m \times n$ matrix $(m < n)$. In this form, linear inequality constraints have been converted to equalities through the addition and subtraction of non-negative 'slack' variables, and incorporated into $Ax = b$. Individual variable constraints, such as $l_j \leqslant x_j \leqslant u_j$, can also be so incorporated, but most computer codes have special features to handle them separately, so as to keep down the dimension of A.

The success of linear programming is due first of all to its usefulness as a model. The accounting world is full of linearity assumptions as to the costs of resources consumed and the value of goods and services produced, so that $c^T x$ is a good approximation to most management objecttives. Many production technologies are also linear, at least to a first approximation, so that portions of A which represent conversion from activity j to resource i need only an estimate of the conversion coefficient a_{ij}; other portions of A usually have large numbers of zero and unity coefficients because of the large number of conservation or 'bookkeeping' relations between different activities – these are intrinsically linear. Finally, upper and lower bounds (especially $l = 0$) on the activity levels, x, are characteristics of our finite world.

The optimal solution to (3.1) is determined almost completely by the constraints. If there are no ties for the optimal solution, x^*, one can show that the optimum is determined by specifying a basic set $\boldsymbol{B}$ of m variables, $x^B = \{x_j | j \in \boldsymbol{B}\}$. The values of these variables are determined by solving the square system

$$Bx^B = b, \tag{3.2}$$

where B is the set of m columns of A corresponding to $\boldsymbol{B}$; these must represent m linear independent vectors in m-space (a basis) so that det $B \neq 0$. These values will, of course, be feasible only if $x^B \geqslant 0$ (and usually $x^B > 0$). The remainder of x^* is determined by setting the $n-m$ non-basic variables, $x^{\bar{B}} = \{x_j | j \notin \boldsymbol{B}\}$, to zero. In geometric terms, selecting the $n-m$ non-basic variables determines a corner, or extreme point of the convex polyhedron of feasible solutions, $\{Ax = b,\ x \geqslant 0\}$. Since there are $\binom{n}{n-m} = \binom{n}{m}$ ways to select a basis (although not all result in a feasible x^B), this characterization of x^* suggests there are a combinatorial number of basic feasible solutions to explore.

However, Dantzig (1963) and his coworkers were able to show that an ascent (descent) method, which proceeds from one extreme point of the constraint space to a better extreme point, is a computationally efficient procedure, taking on the order of $2m-3m$ steps, rather than some combinatorial number related to the number of variables, n. This fortuitous property of the 'simplex method', established through computational experience, is the second reason for its popularity. Even today, there is no satisfactory theory for the rapid convergence of this method; theoretical bounds on the number of iterations required are extremely large, except for a few special problems.

The details of the simplex algorithm are quite straightforward of interpretation, although the actual procedures seem strange at first glance. To avoid continued re-inversion of different $m \times m$ matrices, B^1, B^2, ..., to solve (3.2), successive bases are chosen which differ only in one member; by using the Gauss–Jordan reduction method on the full matrix A (with b adjoined), one can easily check that the current and forthcoming extremal solutions are feasible. To be assured that this move increases (decreases) $c^T x$, the current solution is effectively substituted back into the functional, and local gradients can be read off directly from the new coefficients in front of the non-basic variables. A 'pivot step' (application of Gauss–Jordan reduction to one column of data) then displays the next basic feasible solution.

In economic terms, the choice of a 'good' direction is made by imputing the unit profits c^B of each current basic activity back to m unit prices $y = [y_1, y_2, ..., y_m]$ associated with each constraint by solving the dual system $B^T y = c^B$; the local vector of gradients leading away from the current extreme point is then obtained from $c - A^T y$. This leads to the conceptually elegant theory of duality, in which it is shown that (3.1) is equivalent to another linear program:

$$\min\ (\max)\ b^T y,$$
$$A^T y \geqslant (\leqslant)\ c,$$
$$y \text{ unrestricted},$$

in the sense that, if one program has a finite optimum, then so does the other, and $b^T y^* = c^T x^*$. The dual variables, y, are essentially extended Lagrange multipliers. Optimality is recognized by the 'complementary slackness' condition, $(x^*)^T(A^T y^* - c) = 0$.

There are many different elaborations upon the basis simplex method that purport to solve problems with special data in more efficient ways, or to fully exploit special structure or computational capabilities, or which carry out post-optimal sensitivity analyses. Full details on this by now classical topic may be found in almost any of the approximately 200 texts (Gerber 1974) on linear programming, such as Dantzig (1963), Gass (1958), Simonnard 1966). FORTRAN and ALGOL programs are given in Künzi, Tzschack & Zehnder 1971). See also the comments in Woolsey (1973).

There is a great deal of difference between the classroom representations of linear programs and the actual computer codes (a term to avoid confusion with 'programs') which solve them. An l.p. code must not only accept and convert a variety of input constraints, it must 'get started' (by itself, or from a prior solution), reject unfeasible problems, and carry out a variety of post-optimal sensitivity analyses. Furthermore, as the size of successful l.p. solutions is increased, there has been increasing pressure to further increase the capabilities of advanced programming systems. This means that a great deal of attention must be paid to what we might call the computer science aspects of the program: allocating data between different storage media, and moving it about rapidly; efficient methods of storing inverses of sparse matrices, updating them during the pivot to a new extreme point, and cleansing them of accumulated round-off errors; and finding the best compromise between moving in the direction with steepest gradient or the one with greatest change in $c^T x$. Details may be found in Orchard-Hays (1968), Beale (1967, 1970), (Bonner 1969); Tomlin (1972) and White (1973) have more recent surveys.

For small-scale linear optimization, we are clearly reaching a point of diminishing returns on solution efficiency. A problem with $m = 200$ which cost \$1000 to solve in 1956 can now be turned out for under \$20; most of this improvement is due simply to the generally decreasing cost of digital computation, which trend will no doubt continue. However, the demand for increased computational capabilities continues unsatisfied, as programmers extend the size and boundaries of problem fomulation. Currently, problems with $m = 2000$ are routinely solved, and there are large-scale systems which can handle $m > 10000$; since there is no theory to predict simplex method efficiency, evaluation of new algorithms must follow computational trials. Occasionally there are surprises; for example, Harris (1973) reports a reduction by factors of 2–6 in the number of iterations needed to solve problems with $m = 2$–5000, by using the concept of a fixed datum basis in which to compute and compare gradients.

One way of handling larger size problems to take advantage of any special structure in the constraints. As mentioned above, it is trivial to include individual constraints of the forms $l_j \leqslant x_j \leqslant u_j$. In many scheduling and distribution problems, one encounters constraints of the form

$$l_K \leqslant \sum_{j \in J_K} x_j \leqslant u_K,$$

where the $\{J_K\}$ are non-overlapping subsets of the variables. Problems with a large percentage of constraints of this type have important savings in computer time by using the generalized upper bounding technique of Dantzig & Van Slyke (1967), and recent l.p. codes include this capability.

Large-scale multi-time or multi-sector planning models have highly structured matrices A, with 'block-angular' and 'staircase' non-zero sub-matrices (often with repetitive internal structure), and zero elements elsewhere. For many years it was thought that there might be efficiencies by using the simplex method separately on each subproblem, periodically reconciling the linking constraints in an overall simplex approach; however, in spite of the large literature on these methods (see, for example, Lasdon 1970)), the coding techniques are quite difficult and specialized to the problem structure, and this approach is not now an important contender for these problems.

Another conceptually appealing approach to large-scale l.p. problems which have many separable subproblems linked together by a few 'master' constraints has been the decomposition method of Dantzig & Wolfe (1960). In this method, different values of artificial objective coefficients are 'sent down' to the subproblems, which are solved individually (and hence efficiently), producing a variety of extreme point solutions for each subset of variables; the 'master program' is then solved to satisfy the linking constraints by mixing these extreme-point 'plans' in an optimal way. The dual solution to the master program then produces another set of surrogate goals for the subprograms, which in turn provide other plans for the master optimization. Perhaps the most important feature of this approach is that it quantifies the conditions under which partial or completely decentralized economic planning can take place (Baumol & Fabian 1964). However, as a computational strategy for purely linear programs, decomposition has proven disappointing. It is still a useful approach when the subproblems are not linear programs (see the cutting-stock problem below), are soluble by special procedures (such as transportation problems), or are linearization approaches to nonlinear programming; Lasdon (1970) has several applications of the decomposition approach.

In a recent O.R.S.A./T.I.M.S. Panel (entitled 'is mathematical programming moribund?'), D. Smith described the four eras of mathematical programming. In the 1950s, when the simplex method was new and incompletely understood, the basic problem was to write l.p. codes for the different computers; because of size and speed limitations, emphasis was also placed on development of special-purpose algorithms for special model structures. By the 1960s, fast commercial computer codes were available for production use, and the bottleneck was in the translation of output into the management process; features such as post-optimal sensitivity analysis were added, and more attention paid to simplified data entry, and to summary report generators. As computational capabilities and management sophistication have increased in the 1970s, we find model boundaries expanding, and nonlinear, integer, and decentralized optimization capabilities are being added to the computing systems. Finally, as we head into the 1980s, the routine solution of extremely large problems raises enormous questions of data-base management: how will data be gathered, stored? How will it be checked, cleansed, and updated? Who will certify the results of the optimization and what methods will be used?

4. Nonlinear programming

Optimization problems with nonlinear objective functions and/or nonlinear inequality constraints are of increasing importance in operations research.

The earliest models were of chemical, metallurgical, and hydraulic/gas transmission processes, where the basic transformations are nonlinear, and in engineering design problems (Duffin, Peterson & Zener 1967; Zener 1971). where empirical 'posynomial' forms, $\Pi x_i^{\alpha_i}$, are encountered. Quadratic objectives arise naturally in least-squares approximations (Golub &

Saunders 1970), in electrical networks (Dennis 1959) and control applications (Luenberger 1972) where energy dissipation is minimized, and in location and space assignment problems where Euclidean distance is the measure of optimality. Even with linear economic assumptions the system objective may be profit-per-unit (item, time, trip, etc.), giving a linear fraction, the ratio of two linear forms, to be optimized. In so-called stochastic programming, the objective form includes the expected cost of compensating for the random effect of a decision (Sengupta 1972; Vajda 1972). Nonlinear constraints arise from similar considerations.

In order to retain the desirable property that a local optimum is also a global optimum, one usually restricts investigation to the so-called convex programs, those that can be put in the form:

$$\max\,(\min)\, f(x),$$

$$\left.\begin{aligned} g_j(x) &\geqslant 0 \quad (j = 1, 2, \ldots, p), \\ Ax &= b, \\ x &\geqslant 0, \end{aligned}\right\} \tag{4.1}$$

where $f(x)$ is a concave (convex) function, and each nonlinear constraint is an inequality, formed only from concave functions. This guarantees that the total constraint space is a convex region, and that local search methods will converge, if properly set up, to the overall optimum. Occasionally, non-convex objectives of special form can be handled; otherwise, one must be content with an x^* which may only be locally best.

A glance at recent texts in nonlinear programming (Himmelblau 1972; Luenberger 1973; Avriel 1976) reveals that there is no single preferred method, but rather a variety of different approaches suited to the many different special forms that (4.1) can take. The methods divide themselves naturally into those based strongly on l.p. techniques, those based upon unconstrained optimization algorithms, and specially developed algorithms.

In the first category, we usually have no (or few) nonlinear constraints, and only mildly nonlinear objectives. The first remark is that quadratic programs, where $f(x) = c^Tx + \frac{1}{2}x^TQx$ and Q is a negative (positive) semidefinite matrix, can be solved using variants of the simplex method (Dantzig 1963; Boot 1964). Linear fractional programs, where $f(x) = c^Tx/d^Tx$ is neither convex nor concave, can be handled by treating the denominator as a parametric variable (see, for example, Lasdon 1970). More generally, we can use linear approximations to the nonlinear functions in several different ways. The first approach is to use 'grid linearization', with each nonlinear function recursively defined in terms of local piece-wise linear forms; the complete method is related to the decomposition algorithm, and is especially simple if the nonlinear function is of separable type, $f(x) = \Sigma f_j(x_j)$ (see Lasdon 1970). The more direct 'approximation programming' approach replaces nonlinear functions by their first-order Taylor series approximation expanded at the current solution, $x^{(t)}$; under certain conditions, the optimal solution to the current l.p. leads to a new estimate $x^{(t+1)}$, where another approximation is made, and so on. The (generalized) 'reduced-gradient' algorithm partitions the variables into basic and non-basic sets, as in the simplex algorithm, and consider the optimization only in terms of the latter, thus 'projecting the gradient'. The 'cutting-plane' algorithms turn a problem into a linear objective with nonlinear constraints, and then successively bound or support the desired region by a sequence of hyperplanes. Details and further references on these and other simplex-like methods may be found in Beale (1967), Himmelblau (1972) and Avriel (1976). Most of these algorithms have rather slow convergence since local movement must be restricted to guarantee feasibility or convergence. The exception is the generalized reduced gradient method which seems to remain among the best nonlinear codes (Colville 1970).

Turning now to the methods which are more closely related to unconstrained optimization, we note that, if we have a current solution $x^{(t)}$ which is in the interior of the solution space, we can use a steep ascent method, moving in a sequence of straight-line steps until the optimum or a constraint is reached. If the current solution lies on the boundary of a constraint (or if the original formulation has linear equality constraints), then a (locally or globally) feasible direction must be chosen. In the 'gradient projection' approach, due to J. B. Rosen, the gradient is projected onto the active constraint hyperplanes, and a step is taken in the (reduced) steepest descent direction, until the optimal point along this line is reached, possibly with another constraint becoming active; nonlinear constraints can be handled by approximation, but then extra steps to re-enter the feasible region may be needed. On the other hand, the 'feasible directions' method, due to G. Zoutendyk, determines only directions that are totally feasible, but has difficulty accommodating linear equality constraints. Both of these methods have been considerably elaborated and improved by coupling them with modern methods of steep ascent. A recent survey of the many possible algorithms is in Avriel (1976).

Penalty function methods operate differently, by incorporating the constraints into the functional in various ways, and using unconstrained optimization techniques. Exterior penalty functions add nonlinear costs whenever the solution trajectory leaves the feasible region. In the more popular interior penalty methods, nonlinear 'barriers' are placed in the interior of the feasible region to keep the optimal solution away from the boundaries; these barriers are slowly relaxed so that the sequence of unconstrained optima converges to x^*. For example, if all the constraints are of the form $g_i(x) \geqslant 0$, one would maximize $f(x) - \lambda \Sigma g_i^{-1}(x)$ for a decreasing sequence of λ. This approach has been extensively studied by Fiacco & McCormick (1968), and seem to be the most successful way to tackle problems with strongly nonlinear constraints (McCormick 1971). See also Avriel (1976) and the numerical example in Himmelblau (1972).

An important special class of nonlinear programs are engineering design problems in which the objectives and constraints are composed of posynomials. These problems, referred to as geometric programs, have been extensively analysed, and special-purpose algorithms developed (Duffin *et al.* 1967; Zenner 1971; Beightler & Phillips 1976). It is also possible to find approximate solutions by the simplex method (Ecker & Zoracki 1976).

Optimality in nonlinear programs is recognized by constructing extended Lagrangean multipliers and checking that the so-called 'Kuhn–Tucker conditions' (similar to complementary slackness in linear programs) are satisfied. These multipliers are essentially dual variables and have similar economic interpretations; yet, duality theory does not seem to play a large rôle in nonlinear computations. Part of the difficulty is that there are many different possible dual formulations to nonlinear programs, and, in constrast to l.p., the primal variables, x, also appear in the dual programs, thus preventing their independent solution. The other difficulty is that dual formulations may require delicate analysis; Geoffrion (1972) illustrates some of the difficulties. Convergence properties are also difficult to establish (Wolfe 1970; Zangwill 1969).

Nonlinear programming methods are finally beginning to sort themselves out after a period of diverse theoretical development. What is needed now is extensive computational comparison on large-scale practical problems to further match method to problem; it is probably too much to hope that a universally efficient method will ever be found.

5. Network flow models

If one had to identify the most popular application of linear programming, it would certainly be the network flow models, first investigated systematically by Ford & Fulkerson (1962). In the prototypical problem, we imagine a connected network made up of nodes $i = 1, 2, \ldots, N$, and A directed arcs, each labelled by an ordered pair representing the start and terminal nodes for that arc. Thus, arc (i,j) starts at node i, ends at node j, and we suppose it has flow x_{ij}, limited by upper and lower capacities, l_{ij} and u_{ij}, and contributes a profit (cost) $c_{ij}\, x_{ij}$ to the total operation. The optimization problem is then to find a maximal profit routing of flow:

$$\left.\begin{aligned} &\max\,(\min) \sum_i \sum_j c_{ij}\, x_{ij}, \\ &\sum_j (x_{ij} - x_{ji}) = q_i \quad (i = 1, 2, \ldots, N), \\ &l_{ij} \leqslant x_{ij} \leqslant u_{ij}. \end{aligned}\right\} \tag{5.1}$$

Here $q_i > 0$ [$q_i < 0$] represents external flow into [out of] the network at node i; the equality constraints represent 'Kirchoff law' conservation at each node – summations are understood to be only for arcs actually connected there.

(5.1) is clearly a linear program, but of very special structure, since the constraint matrix only contains 0's, +1's, and −1's, and exactly one of each of the latter for each variable. One can show that this implies:

(1) There is exactly one redundant conservation equation, and for solvability $\Sigma q_i = 0$;

(2) The optimal solution is obtained by adding and subtracting the boundary flows $\{q_i\}$, and is thus integral if they and the capacities $\{l_{ij}, u_{ij}\}$ are integral;

(3) The l.p. basic solution is related to a network configuration called a tree – a subset of $N-1$ arcs which connects all the nodes, and has no loops;

(4) 'Pivoting' from one extreme point solution to another is related to passing flow around a loop in the network, so the only computation labour is to find a 'good' loop and stay feasible.

Because of this simplicity in the optimal solution, it has been possible to develop fast, special-purpose computer codes which can handle much larger network formulations than could be handled by a general l.p. code – 10^4 nodes and 10^6 arcs being handled routinely (Glover & Klingman 1975).

(5.1) includes a variety of very useful simpler models. For example, if the network consists only of all possible links between one set of nodes with $q_i > 0$ (the plants) and another set of nodes with $q_i > 0$ (the customer), we have the classical 'transportation problem'; making the two sets equal in size and all $q_i = \pm 1$ further reduces (5.1) to the 'assignment problem'. For a general network, if in (5.1) $c_{ij} = 1$ for a certain arc (i,j), zero for all others, then we have the 'maximum (minimum)-flow problem', which can be solved by a simple 'labelling method' (Ford & Fulkerson 1962); conversely, if all $q_i \equiv 0$, but $l_{ij} = u_{ij} = 1$ for a certain arc (i,j), then we have the 'longest (shortest)-route problem' for which a variety of special-purpose algorithms are available (Dreyfus 1969). In fact, most algorithms for the general network problems are alternate applications of max-flow and shortest-route procedures to different arcs in the network. Easy modelling extensions include dynamic flows, capacities on nodes, parametric studies, etc. (Fulkerson 1966; Price 1971).

The literature is full of a number of confusing references to 'primal-dual' ['Ford–Fulkerson'], 'dual', 'simplex', and 'out-of-kilter' methods. These are, in fact, historical variants of the same extreme point method which differ only in starting procedures, selection of new variables to enter the basis, treatment of initial infeasibilities, etc. Nevertheless, there seems to be considerable difference in computational efficiency of the different approaches, and upon the manner in which labelling information is stored; see the studies of Glover, Karney, Klingman & Napier (1974*a*, *b*) and Glover & Klingman (1975). This is important because of the many general optimization models which have network-flow substructures.

Incidentally, linear network models differ from general linear programs in that reasonable bounds on the number of iterations can be obtained (Edmonds & Karp 1972; Dreyfus 1969). Zadeh (1973*a*, *b*) gives some worst-case examples.

The linear program dual to (5.1) has constraints of the form $y_i - y_j \geqslant (\leqslant)\ c_{ij}$, where the dual variables $\{y_i\}$ have an interesting physical interpretation as node potentials, obeying Kirchoff's potential law, $y_i - y_j = c_{ij}$, for arcs in the optimal basic solution. This dual program also arises directly in the critical-path scheduling models (see below).

Because of the success of the initial special-purpose algorithms, several extensions of flow models have been proposed, with as much of the 'on the network' simplicity retained as possible. For example, in 'networks with gains', we imagine that each arc (i, j) has a multiplier k_{ij} which converts the incoming flow x_{ij} to an output flow $k_{ij}\,x_{ij}$; this formulation includes a variety of interesting new applications (Jewell 1962), but requires complicated labelling schemes, since the solutions are not integral, and conservation in-the-large is not satisfied. Maurras (1972) reports on recent computational experience; Glover & Klingman (1973) show that some networks with multipliers can, in fact, be reduced by scaling to ordinary networks, and Truemper (1976) discusses scaling in general.

Another extension studied in great detail is the multicommodity flow problem, in which several types of flow pass without mixing over the network to satisfy their own boundary requirements, but are mutually constrained by each arc's total flow capacity (Ford & Fulkerson 1958; Jewell 1966). Even with integral constraints, the optimal answer may require fractional allocations of capacity. Models of this type are important in communication networks (Frank & Frisch 1971) and road traffic problems (Potts & Oliver 1972). Recent algorithms are given by Hartman & Lasdon (1972) and, with computational experience, by Grigoriadis & White (1972).

It should be mentioned that quadratic profit (or loss) on arcs can easily be handled; the procedures are a mixture of linear flow procedures and the methods of electrical circuit theory (Dennis 1959).

A variety of other extensions to network flows have been proposed, but, generally, the days of developing special-purpose algorithms are over, due to the rapidly increasing capabilities of all-purpose mathematical programming codes. Networks remain, however, a fruitful area of research for other types of models, such as stochastic flows (Frank & Frisch 1971; Kleinrock 1976) and various combinatorial routing and covering problems (see below). Bellman, Cooke & Lockett (1970) give some other clever network problems.

Also in the network flow class of l.ps are the critical-path scheduling problems (Moder & Phillips 1970). In this model, the network represents the precedence relationships between the different jobs of a project. The decision variables are dates $\{y_i\}$ on each node, such that for (i, j) there is sufficient time to complete job (i, j), requiring time t_{ij}; i.e. $y_i \geqslant t_{ij}$. The objective

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is to minimize total elapsed time on the project. Since this is exactly the dual program to a longest-route problem, the solution procedures are trivial; however, the model has found widespread utility in the construction industry.

6. Integer linear programming

In many optimization problems the assumption of a continuous decision variable is untenable and one would like an integral answer for some or all of the variables; for instance, only an integer number of round trips can be made by vehicles, and integral numbers of spare parts stocked for space voyages. Variables which can only take on values 0 or 1 are particularly useful in modelling selection processes with an attendant fixed cost; for example, if activity j costs nothing if not selected, and costs $d_j + c_j x_j$ when operated at level x_j, $0 > x_j \leqslant u_j$, then we can formulate it as part of a ('mixed') integer program with a cost $d_j z_j + c_j x_j$, and constraints $0 \leqslant x_j \leqslant u_j z_j$, $z_j \in \{0, 1\}$. The $\{z_j\}$ might themselves be jointly constrained (see Balinski 1965; Garfinkel & Nemhauser 1972*a*); other surveys of the field are Balinski & Spielberg (1969), Greenberg (1971), Garfinkel & Nemhauer (1972*b*).

The earliest attempts to solve integer linear programs were based on the idea of rounding-off the variables in the corresponding continuous linear program; however, these failed because it is easy to construct examples where the optimal integer answer is not the feasible integer point nearest to the best l.p. solution – in fact, can be arbitrarily far away. Only after the initial paper of Gomory (1958) were exact solutions possible. His idea was that of adding additional constraints to sequentially generate the convex hull of feasible integer points; these 'cuts' remove part of the original solution space but do not remove any integer solutions. After again optimizing using the simplex method, the new non-integral x^* suggests other cuts, and so forth. Many different methods of generating these cutting planes are now available, but, generally these methods have slow convergence, and have been tested mostly on problems of less than a hundred integer variables.

The most efficient methods for general integer programs are currently based upon implicit enumeration techniques – the so-called 'branch-and-bound' method (Agin 1966; Lawler & Wood 1966; Mitten 1970) or 'progressive-separation and evaluation' procedure (Bertier & Roy 1964; Roy, Benayoun & Tergny 1970). The basic idea can be illustrated by considering the integer program:

$$\left.\begin{aligned} \max P &= c^T x, \\ Ax &= b, \\ x_j &\in \{0, 1, 2, \ldots, u_j\} \quad (j = 1, 2, \ldots, n). \end{aligned}\right\} \tag{6.1}$$

An upper bound on $P^* = c^T x^*$ can obviously be obtained by solving the corresponding l.p. with the integer constraints replaced by $0 \leqslant x_j \leqslant u_j$ $(j = 1, 2, \ldots, n)$. Now, pick a certain variable to 'arbitrate' or 'branch upon' – say x_1. Problem (6.1) is 'separated' into $u_1 + 1$ distinct integer programs in which x_1 is *fixed* at its possible values $0, 1, 2, \ldots, u_1$; the objective function for each of these subproblems can then be bounded from above by a linear program in which $0 \leqslant x_j \leqslant u_j$ $(j = 2, 3, \ldots, n)$ – and these solutions are usually within a few simplex steps of each other, as x_1 is varied parametrically through its integer values. To proceed, the 'best' choice of x_1, in terms of the various bounds on P, is taken, and a different variable is chosen for further exploration, generating a new sequence of subproblems in which two variables are now fixed at integer values. At successive steps, one takes the 'best' overall candidate solution, irrespective of the number of arbitrated variables. Under very general conditions, one can show that this procedure terminates when the first solution with *all* variables fixed is obtained. This enumerative approach could, in principle, require exploration of all possible $\Pi\ (u_j+1)$ solutions, but in practice behaves rather well if good rules for the selection of successive variables are used. A convenient framework for explaining the various approaches is in Geoffrion & Marsten (1972). Commercial mixed-integer codes are now undergoing extensive computational testing, with encouraging results for problems with several hundred integer variables and several thousand constraints (Roy *et al.* 1970; Benichon *et al.* 1971; Mitra 1973; Forrest, Hirst & Tomlin 1974). It is interesting that the most successful solutions use a number of heuristic procedures, and depend strongly on the way in which the problem is formulated (Geoffrion 1976).

Naturally there are speedier special-purpose algorithms available for specific models, for instance, if the l.ps used to determine the bounds are of the network flow type (Balinski & Spielberg 1969). Geoffrion & Graves (1974) report a successful application of an older method due to Benders (1962) to a very large warehouse-location and multicommodity distribution problem. Other special-purpose methods and models are described below.

Finally, Kalvaitis & Posgay (1974) describe a very successful commercial application of integer programming, while Woolsey (1972) injects a cautionary note.

7. Combinatorial optimization

The boundary between integer programming and combinatorics is not a precise one, since many of the problems we consider here have formulations like (6.1), with 0 or ± 1 constraint coefficients. However, combinatoric problems are generally more 'puzzle-like', are either very simple or very difficult, and usually require special algorithmic development. The most important ones are related to network applications.

To give some examples, imagine a network (or, if you prefer, a connected graph) with A undirected arcs and N nodes. Each arc has a positive unit cost; the total cost of 'using' a certain subset of arcs is the sum of the unit costs associated with those arcs. Now consider the following different problems:

(1) Given two specific nodes, find the least-cost (shortest) path (sequence of arcs with nodes in common) from one node to the other;

(2) repeat (1), but find the 2nd, 3rd, ..., kth shortest paths;

(3) find the least cost subset of arcs which will connect all nodes to each other;

(4) find the minimal-cost tour which passes through all nodes at least once, returning to the starting node;

(5) find the minimal-cost tour which traverses each arc at least once.

The shortest-path problem (trivially extended to undirected arcs) has already been discussed as a special linear program of the network flow type; it can be solved in the order of N^2 steps using a special dynamic-programming-type algorithm; problem 2 also has an efficient dynamic-programming formulation (Dreyfus 1969).

In problem 3, one can easily show that the desired configuration is a (spanning) tree, $N-1$ arcs which connect all nodes, and has no loops. This problem cannot be posed as an l.p., and yet is susceptible to almost any kind of 'greedy' heuristic: for example, pick the least-cost arc, then add to it the next-lowest-cost arc not forming a loop, then add to those the next-lowest-cost arc not forming a loop, ... and so on, until a tree is obtained in at most A steps. This model is useful in a variety of communication problems (Frank & Frisch 1971; Pierce 1975).

Problem number 4 is the well-known 'travelling salesman' problem, and is fundamentally more difficult than 3. Certain general theorems are known (Bellmore & Nemhauser 1968); for example, if the unit costs obey a 'triangle inequality' (it is always cheaper to go from one city to another in one step than in two), then the optimal tour is a 'Hamiltonian cycle' – a circuit of N arcs visiting each city once and only once. A variety of different approaches have been proposed for this problem: dynamic programming, where storage bottlenecks limit the size of the problem; integer linear programming formulations, which required the addition of $2^{N-1}-1$ constraints – later improved by cutting-plane algorithms; and branch-and-bound algorithms which have variable performance depending on the heuristics chosen. Bellmore & Nemhauser (1968) provide a summary; branch-and-bounding appears already as the best method, but only problems with $N < 100$ were solved exactly. Held & Karp (1970, 1971) have found that sharper bounds, derived from solving a related spanning tree problem, can produce important efficiencies. Their approach has been further improved by Hansen & Krarup (1974), and, for directed arc networks, by Smith, Srinivasan & Thompson (1975); computation times seem to vary about as $N^{3.5}$ for small problems, but $N = 200$ is about the limit for exact solutions. Webb (1971) and Lim & Kernighan (1973) show how to obtain good approximate solutions for larger problems.

Problem number 5 is called the 'Chinese postman' problem. In undirected arc networks where every node has an even number of arcs, there exists an 'Euler tour', a tour which passes through each arc only once; this is then optimal. In the contrary case, extra trips are necessary to pass through nodes of odd degree; this is done through an associated integer 'matching' problem for which good computational experience is available (Edmonds & Johnson 1973).

Both the travelling-salesman and Chinese-postman problems are important as building blocks in realistic routing applications. Orloff (1974*a*) has synthesized these methods to solve a general routing problem (in which the minimum-cost tour is to visit a subset of the nodes and cover a subset of the arcs); this has important application to the problem of routing a fleet of vehicles out of a central facility, as in school-bus-routing, and refuse scavenging (Orloff 1974*b*; see also Bennett & Gazis 1972; Beltrami & Bodin 1974, and the survey in Bodin 1975).

The concept of choosing an optimal tour to pass through certain nodes or arcs of a network can be generalized to the combinatorial problems of 'set covering', in which costly subsets are to be chosen (from a given family of subsets) so as to span the original set of elements at minimal total cost; if the selected subsets are also to be disjoint, it becomes a problem of 'set partitioning'. See Garfinkel & Nemhauser (1972*b*, *c*) and Balas & Padberg (1976) for surveys of methods and references; Marsten (1974) reports recent computational experience.

These set covering/partitioning models can be applied to a variety of discrete selection problems. Perhaps their most useful application to date has been to the problem of scheduling airline crews to 'cover' a flight schedule at minimal salary, living-expense, and 'deadheading' costs, subject to various restrictions on work-and-rest times, company and union requirements, etc. The total problem is quite complex, and a variety of approaches have been proposed at intra-industry meetings; a convenient summary is in Arabeyre, Fearnley, Sterger & Teather (1969).

Another highly visible use of integer programming is in the problem of determining political districts so as to achieve equity in terms of absolute deviations of distinct population from an overall mean. Minimizing the sum of such deviations over all districts is a set partitioning problem; if the objective is to minimize the largest of such deviations, we have a 'bottleneck problem' for which a branch-and-bound procedure has been tested on a state districting problem with 40 indivisible population units (Garfinkel & Nemhauser 1970). A related political topic is the problem of determining a fair apportionment of representatitives between political units; Balinski & Young have developed a new 'quota method' which uses integer programming ideas, and have applied it to the U.S. Congress (1975) and the European Parliament (1976).

Finally, no discussion of combinatorics would be complete without a discussion of new results on computational complexity. There are certain problems, such as maximal-flow, shortest-route, assignment, and minimal spanning-tree problems, where one can guarantee that the number of solution steps is less than some polynomial function of the problem parameters; i.e. a 'polynomial-time-bounded algorithm' exists (Edmonds 1965). Using modern concepts of algorithmic analysis (see for example, Aho, Hopcraft & Ullman 1975), Karp has shown that a variety of other combinatorial problems are equivalent in the sense that if any one of them can be solved, then there is a polynomial-time-bounded transformation which will solve any of the others; if follows then that all or none of the members of this 'NP complete class' are solvable in polynomial time (Karp 1972, 1975). This class is quite wide and contains the travelling-salesman, integer-linear programming, knapsack problems, and set covering and partitioning problems; since all of these are computationally difficult, we suspect that none of this is polynomial-time-bounded. Of course, this does not mean that there cannot be efficient algorithms for moderate-size problems, or even that an 'average' problem cannot be solved during a time which is a polynomial function of its size, as we have seen. In fact, many heuristic algorithms have already been surprisingly successful in solving actual combinatorial problems. An exciting new line of research is now trying to quantify this success by looking at the proximate success of heuristics on *distributions* of problem parameters; in many cases one can guarantee that all but small percentage of such problems will be optimally solved by a fast algorithm (Karp 1976).

8. Dynamic programming

Dynamic programming is not so much a method of optimization as it is a framework in which to efficiently analyse loosely-coupled, repetitive decision problems. Typically, these problems arise in dynamic models (or serial processes) where the decisions made at one instant (or stage) give rise to a similar problem at a later time (the next stage), in which a parameter or the state of the system has changed.

A typical model is the resource allocation problem, in which activity j, operating at level x_j restricted to some set of values S_j, uses up an amount $a_j > 0$ of resource, and generates revenue $r_j(x_j)$, $(j = 1, 2, \ldots, N)$. Assuming there are B total units of resource, the global optimization problem is:

$$\left.\begin{aligned} &\max \sum_{j=1}^{N} r_j(x_j), \\ &\sum_{j=1}^{N} a_j x_j \leqslant B, \\ &x_j \in S_j \quad (j = 1, 2, \ldots, N). \end{aligned}\right\} \tag{8.1}$$

If the return functions are linear, and S_j is an interval on the real line, this is a simple l.p.; more general problems require special handling, even with just one constraint. In the dynamic programming approach, we solve the problem in stages, usually beginning with the last. Let $f_N(b)$ be the optimal return from the Nth activity, assuming b units of resource are made available to it; this is obtained from the simple optimization:

$$\left.\begin{aligned} f_N(b) &= \max r_n(x_N), \\ a_N x_N &\leqslant b, \\ x_N &\in S_N, \end{aligned}\right\} \tag{8.2}$$

which is solved parametrically, for all values of $0 \leqslant b \leqslant B$. Knowing $f_N(b)$ (and having recorded the optimal decision $x_N^*(b)$), we now proceed to the determination of $f_{N-1}(b)$ – the optimal return from the last two activities, assuming b units of resource are available for *both*. Since activity $N-1$ uses up a_{N-1} of this resource, we have again a one-dimensional optimization:

$$\left.\begin{aligned} f_{N-1}(b) &= \max r_{N-1}(x_{N-1}) + f_N(b - a_{N-1}\, x_{N-1}), \\ a_{N-1}\, x_{N-1} &\leqslant b, \\ x_{N-1} &\in S_{N-1}, \end{aligned}\right\} \tag{8.3}$$

which is solved parametrically for $0 \leqslant b \leqslant B$. This process is carried out successively for all preceding stages until solving $f_1(b)$ for $b = B$ gives the optimal total return. Note that the optimization difficulty is reduced to that of a one-dimensional problem, but that, in exchange, a sequence of optimal returns (and policies) must be stored for all values of b. Thus, large dynamic programs are typically storage-limited, and it is difficult to adequately handle problems with more than 2 or 3 linking constraints. In other formulations, the result of one stage's optimization is to leave the system in a different abstract state (such as location and position in space), rather than with a diminished scalar variable; here the 'curse of dimensionality' requires one to quantize the state space rather grossly to get an initial approximation in reasonable computation storage, and then to successively refine the space (Larson 1968). Bertelé & Brioschi (1972) have analysed non-serial models which require a different kind of successive approximation. Dirickx & Jennergren (1975) examine myopic policies.

The observation that many dynamic models could be reduced to a series of one-stage parametric problems is due to R. Bellman, who called it an optimality principle: 'An optimal policy has the property that, whatever the initial state and initial decision are, the remaining decision must constitute an optimal policy with regard to the state resulting from the first decision' (Bellman 1957). A variety of different formulations which use this principle can be found in the still-useful texts (Bellman 1957, and Bellman & Dreyfus 1962).

Despite their inherent limitations, dynamic programs are extremely useful for problems of moderate size, or as subroutines in larger problems. The premier example of this is the cutting stock or trim problem, thoroughly investigated by Gilmore & Gomory (1961, 1963, 1965, 1966). Imagine that an order for different numbers of different lengths is to be filled by cutting from a number of larger, standard lengths; the problem is to minimize wastage of stock. If the size of the order is large, an approximate solution can be found by using a linear program in which the variables are all the different possible patterns of cutting the standard lengths. Instead of enumerating (the combinatorial number of) all such patterns, Gilmore & Gomory generate new candidate patterns by solving a related 'knapsack problem', which is the name given to (8.1) when the returns are linear, $r_j(x_j) = r_j . x_j$, and the variables are integer, $S_j = \{0, 1, 2, \dots, u_j\}$. In this way, they alternate between linear and dynamic programs to solve a complex problem with many industrial applications.

The knapsack problem is an important model in its own right, both because of its usefulness as a building-block in realistic packing situations, and because of its deceptive simplicity. For example, if the activities (items to be selected for the knapsack) are arranged in decreasing order $r_1/a_1 > r_2/a_2 > \ldots$ (decreasing value per unit resource (space or weight) consumed), and if x_j can be fractional, $0 \leqslant x_j \leqslant u_j$, then the solution is trivial: take $x_1 = \min(u_1, B/a_1)$; $x_2 = \min(u_2, (B-a_1 x_1)/a_2)$; etc. It might be thought that this 'greedy' algorithm might also extend to the integer case by making the obvious modification in the 2nd term; unfortunately, this is only true for very special data sets (Magazine, Nemhauser & Trotter 1973), and we know that the general knapsack problem belongs to the potentially difficult NP-complete class of problems. Many other integer programming methods, such as cutting-plane and branch-and-bound algorithms, have been proposed (Garfinkel & Nemhauser 1972*a*).

Dynamic programming is also useful as a theoretical tool, in proving the optimality of certain *forms* of decision, rather than assuming the form, and merely setting the control parameters optimally. Perhaps the most important results of this kind was the proof that the two-bin inventory control policy was optimal under certain cost and demand assumptions (Arrow, Karlin & Scarf 1958). Also, one can often show that dynamic optimization problems in operations research have a limiting behaviour, in the sense that, as the planning horizon increases without limit, the optimal total return may be bounded in value (if returns discounted over time) or be bounded by a linear function of time (if undiscounted), and the optimal decision may be stationary; the optimality principle then becomes a recursion relation which can be solved iteratively. A well-studied example of this is the model of Markov programming, in which a decision z_t take a process in state i at time t into state j at time $t+1$ with probability $p_{ij}(z_t)$, earning a reward $r_{ij}(z_t)$, $(i, j = 1, 2, \ldots, N)$ $(t = 0, 1, 2, \ldots)$; thus, if the policy is fixed for all t, the process follows a Markov chain, and the mean total reward is asymptotically proportional to t. Many interesting decision problems can be formulated in this framework (Howard 1960) and there are numerous extensions (Jewell 1963; White 1969; Mine & Osaki 1970). If z_t is a discrete control action and N is finite, this problem can be solved a number of different ways, including linear programming. Problems with continuous decision and state spaces require a certain amount of delicate analysis, and here we must say that the theory has far outstripped the applications (Porteus 1975).

Interesting applications of dynamic programming are appearing regularly in the literature, particularly in various investment and consumption (Hakansson 1970), allocation (Derman, Lieberman & Ross 1975), and 'stopping' (Leonardz 1973) models.

9. Control theory

There is an intimate relation between the theory of dynamic programming and recent developments in control theory, which may loosely be described as optimization of a system of differential (or difference) equations. A survey of this area would take many more pages, and we content outselves with references to Bellman (1967, 1971) for a dynamic programming presentation, and to Canon, Cullum & Polak (1970) and Luenberger (1972) for discussion of the interaction with mathematical programming.

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10. Multiple objectives and decision-making under uncertainty

Analysts have long realized that optimization of a single fixed objective is not responsive to the needs of decision-making in the real world, where many different conditions influence choice. It is of course possible to explore several different objectives simultaneously to evaluate trade-offs; this is especially easy in linear programming with only a few conflicting objectives (Zeleny 1974). However, the general problem is quite difficult, since the decision-maker may not be able to define his preference space precisely until faced with actual comparisons. A variety of different models may be found in Cochrane & Zeleny (1973); Roy (1971) is a synthesis of the different methodologies.

Somewhat the same problem arises in decision-making under uncertainty (Hertz 1973). In some instances, one can justify a single objective, such as maximizing a mean value or the probability of gaining a fixed sum. Usually, however, several aspects of the distribution of outcome seem important; for example, in the $E-V$ approach of Markowitz (1959), one examines the trade-off between mean and variance. More generally, the problem is one of comparing two distributions. Some results can be obtained through ideas of stochastic dominance, but the preferred approach seems to be through utility theory, as developed by Von Neumann & Morgenstern. Given a choice between several distributions of random outcomes $\{p_i(x)\}$, their result states that, given three reasonable hypotheses which a 'rational economic man' might follow when constructing preferences among these gambles, the preferences can always be represented in terms of a non-decreasing utility function, $u(x)$, idiosynchratic to the decision-maker, by ranking the distributions according to the expected utility of the i^{th} gamble,

$$U_i = \int u(x)\, p_i(x)\, \mathrm{d}x.$$

This approach is well explained in Borch (1968); White (1969) and Raiffa (1970). Although there have been numerous objections to utility theory (for example, an $E-V$ decision-maker does not satisfy the hypotheses of the theory), it seems very difficult to modify. And, it does satisfactorily explain certain observed behaviour, such as paying a premium over the mean loss (profit) for insurance (a lottery ticket) if one is risk-averse (risk-seeking). There has been a great deal of attention to the problem of defining multi-attribute preferences, and the construction of an overall utility function; Keeney & Raiffa (1976) is the definitive text.

Utility theory finds usage in the models of 'decision analysis', a new guise for statistical decision theory which emphasizes the formal process of laying out a decision tree, the estimation of the probabilities associated with nature's plays, the estimation of the utility of the terminal outcomes, and the use of Bayes's law and dynamic programming to calculate optimal strategies (Raiffa 1970). Decision analysis is particularly useful as a pedagogical framework, and as means of structuring communication between the analyst and the decision-maker. Suggestive applications appear in Grayson (1960 – drilling decisions); de Neufville & Kenney (1972 – airport development); and Hax & Wiig (1976 – capital investment).

11. Stochastic processes and models

Much of the early literature in O.R. was devoted to the study of random processes, both because of rapid developments in the 1950's, in communications theory, and also because it was not usual to be trained in this area. Now the situation is reversed; we take for granted that the

O.R. specialist has had at least two courses in stochastic processes, and there seem to be few useful new theoretical developments. Çinlar (1975*b*) is an example of a modern theoretical text.

The most ubiquitous model is, of course, the Markov process, especially in its discrete-time, discrete state-space version, the Markov chain. In about every field of application, one can find a Markov chain, possibly imbedded in a more complicated process (Kendall 1953), used to describe successive transitions between states. In a certain sense, it represents the first order of dependence up from a purely independent-transition process, and its modelling success is due to the fact that higher order dependencies are rarely needed. The properties of Markov chains have been well understood for twenty years, thanks to the still excellent book by Feller (1967).

The other useful model is the renewal process, which describes point processes as generated by independent, identically distributed random intervals; this is an obvious model in reliability, where failed items are immediately replaced by new, similar items, but is also useful as a model for other processes, such as arrival of customers at a queue. Cox (1962) is a good introduction to the field; the full generality of renewal arguments and the various limit theorems are covered in Feller (1971).

By combining Markov chains and renewal theory (so that a transition between two states i and j takes a random duration sampled from a distribution depending on i and j), we obtain the very useful Markov-renewal (or semi-Markov) processes. The theory is only mildly more complicated, and subsumes many early elaborations on the basic processes. Çinlar (1975*a*) surveys the field; Teugels (1976) is a bibliography.

The other important extension to these basic theories is the addition of economic functions, called rewards or potentials, for use in optimization. For example, if a transition between states i and j took time t, we might generate a profit $r_{ij}(t)$ at the end of the interval, and add it to other rewards earned from previous transitions. The mathematical details are easy, and are already being included in introductory texts (Ross 1970).

Other stochastic topics, such as random walks, branching processes and diffusion processes find special uses, particularly in queueing theory (Gaver 1968; Newell 1971), and in attempts to model stock market behaviour (Fama 1970), but few other applications.

12. Queueing theory

The study of congestion in service systems was very popular in the 1950s and 1960s. Although the basic modelling had been carried out many years previously by A. K. Erlang, and others, for problems of telephone traffic, the subsequent development of queueing theory showed the essential similarity between congestion and waiting-time phenomena in such diverse applications as road traffic control, inventory management, delays at toll booths, health care appointment systems, machine servicing, water reservoir control, airport scheduling, etc. In an extensive bibliography, Saaty (1966) claims that there are by 1966 over 2000 references in queueing theory, and comments that real improvements in managing congestion phenomena do not match the congestion caused by the number of theoretical papers on the subject; Lee (1966) is also pessimistic. Bhat (1969) refutes these arguments in a convenient summary of the field, and gives a more selective bibliography.

In the prototypical queueing problem we imagine that customers 1, 2, 3, ..., n, ... arrive at a service system at points in time t_1, t_1+t_2, $t_1+t_2+t_3$, ..., $(t_1+t_2+\ldots t_n)$, ..., and queue up in front of a single server, who will process them individually, taking $s_1, s_2, s_n, \ldots, s_3, \ldots$ units of time. We additionally specify 'FIFO' (first-in, first-out) service priority and assume the server begins work as soon as a customer arrives. If we let w_n be the waiting time in the queue of the n^{th} customer and assume that the first customer arrives when the server is idle, we find:

$$\left.\begin{aligned} w_1 &= 0, \\ w_2 &= \max\,(s_1 - t_2, 0), \\ w_3 &= \max\,(w_2 + s_2 - t_3, 0), \\ w_n &= \max\,(w_{n-1} + s_{n-1} - t_n, 0), \end{aligned}\right\} \tag{12.1}$$

One can also describe, for example, the number in the system found by the n^{th} arrival. Equations similar to these could, in principle, be found for other variations in service priority, if the number of servers were increase to m (in parallel), if there were serial stages of queues (with or without intermediate queues), if service was in batches, etc., etc.

The typical analytic assumptions about the input and service processes are that the arrival spacings $\{t_1, t_2, t_3, \ldots, t_n, \ldots\}$ are independent and identically distributed random variables (i.i.d.r.vs) (thus arrivals constitute a renewal process), and service times $\{s_1, s_2, s_3, \ldots, s_n, \ldots\}$ are also i.i.d.r.vs, with a different distribution. In spite of the simplicity of formulation of this so called '$G/G/1$' queue, only a few general results are known:

(1) Statistical equilibrium is achieved if and only if the utlization factor, $\rho = \bar{s}/\bar{t}$ ($\bar{s}, \bar{t}$ – mean service and inter-arrival times), is strictly less than unity;

(2) The fraction of time the server is idle is $1-\rho$;

(3) The customer-average mean waiting time in queue, $\bar{w}$, and the time-average mean number of customers in the queue, $\bar{q}$, are related by:

$$\bar{q} = \bar{w}/\bar{t}.$$

Equivalents of these results hold under more general conditions, for example, if there are m parallel servers, or if a different priority scheme is used. (3) is a very general result which essentially defines what we mean by customer-average wait and time-average queue (Little 1961; Jewell 1967; Eilon 1969; Maxwell 1970). Extensions are Stidham (1972) and Brumelle (1972).

Further general results seem very difficult. The basic problem is that n^{th} delay content, $u_n = s_n - t_{n+1}$, is a two-sided random variable; as long as the partial sums $u_1, u_1+u_2, \ldots$ are positive, they are identically $w_2, w_3, \ldots$. However (with probability one, if $\rho < 1$), for some k, $u_1+u_2+\ldots u_k$ will be negative, $w_{k+1} = 0$, and the process starts over. The analytic determination of the distribution of k (the number served during a busy period) and $i = |u_1+u_2+\ldots u_k|$ (the length of the next idle period) seems very difficult in the general case; but, if they could be determined, we could obtain $G/G/I$ results, such as:

$$\bar{w} = \frac{\sigma_t^2 + \sigma_s^2 + (\bar{t})^2\,(1-\rho)^2}{2\bar{t}(1-\rho)} - \frac{\overline{i^2}}{2\bar{i}}, \tag{12.2}$$

where σ_t^2 and σ_s^2 are variance of the inter-arrival and service r.vs, and $\bar{i}$, $\overline{i^2}$ the first two moments of idle time (Marshall 1968a). Alternatively, we must find the distribution of w from a Wiener–Hopf integral equation.

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The most popular historical way around these difficulties has been to use exponential inter-arrival (Poisson arrival) and exponential service distribution assumptions. Because of the 'memoryless' properties of the exponential, $Pr\{x > x_0+h \mid x > x_0\} = Pr\{x > h\}$, every interval of time is a regeneration point, and queueing systems can be described in terms of continuous-time Markov processes, and solved by linear systems of first-order 'birth-and-death' differential equations. This approach enables us to model unlimited variations, such as different queue disciplines and priorities, balking and reneging, interrupted, blocked, and controlled service, serial and parallel stages, etc. Arbitrary distributions can be approximated through sums or mixtures of random variables. Morse (1958) has many useful models of this type.

Kendall (1951, 1953) was the first to show that only one of the inter-arrival or service distributions need be exponential to complete the analysis. For example, in the case of Poisson arrivals, i is distributed as t, $\sigma_t^2 = (\bar{t})^2$, and (12.2) reduces to the Pollaczek-Khintchine formula:

$$\bar{w} = \frac{\sigma_s^2+(\bar{s})^2}{2\bar{t}(1-\rho)}. \tag{12.3}$$

If service times are exponential, one can analyse an embedded Markov chain, even with m servers (see, for example, Kleinrock 1976). Unfortunately, the assumptions of exponentiality led naturally to the use of transform methods, and the papers of the 1960s are overburdened with the machinery of LaPlace and Fourier.

Some of the most interesting recent research in queueing theory has been in the area of approximate and bounding results, especially to the moments of the waiting-time distribution. Kingman (1962*b*) showed that (12.2) has the strict upper bound

$$\bar{w} \leqslant \frac{\sigma_t^2+\sigma_s^2}{2\bar{t}(1-\rho)} = W_u \tag{12.4}$$

for all GI/G/1 queues, and that this bound is a good approximation for $\bar{w}$ in heavy traffic ($\rho \rightarrow 1$), when w is then approximately exponentially distributed (Kingman 1962*a*).

Finding a strict lower bound in terms of moments is more difficult. Marshall (1968*a*) makes some additional shape assumptions on the distribution of t and gets sharp lower bounds. For example, if $E\{x-x_0 \mid x > x_0\}$ is decreasing in x_0 (decreasing mean wait for the next customer as a function of clock time since the last arrival), he obtains

$$W_u - \frac{\bar{t}(1+\rho)}{2} \leqslant \bar{w},$$

which bounds $\bar{q}$ to within one customer! Other single-server variations are in Marshall (1968*b*).

Bounds for the G/G/m queue are given by Kingman (1970) and Brumelle (1971), and have been improved for certain special cases; a convenient summary is in Kleinrock (1976). Finally, we should mention the large amount of unpublished work on approximations for single- and multi-server Poisson queues (Marchal, Gross & Harris 1974; Nozaki & Ross 1976; Takahashi 1976), and the growing interest in estimators (Law 1975).

A valid criticism of all the above models is that they are only useful in stable régimes; time-varying parameters and transient response are difficult to analyse, except in the simplest systems. However, a growing literature is queueing theory begins by approximating the arrival and departure processes themselves, making first a deterministic 'fluid' approximation to the average values of these processes, and then adding a second-order 'diffusion' approximation.

Gaver (1968) has investigated diffusion approximations to the M/G/1 queue, and Newell (1968, 1971) has written extensively on 'rush hour' traffic, when the system is overloaded for a period of time, and then recovers. Although the concepts are simple, the analysis leads to Fokker–Planck diffusion equations, and requires care in arguing the limiting approximations. Kleinrock (1976) contains a clear survey of diffusion models; Whitt (1974) covers recent contributions to limit theorems.

Perhaps the most interesting new application of queueing theory has been in the field of computer time-sharing systems. Kleinrock (1976) contains an excellent description of the analytic and heuristic models developed to analyse multi-user priority schemes and design computer communication networks. The challenge of working with a complicated real network (ARPANET) has clearly provided a fruitful interaction between queueing theory and practice.

13. Reliability theory

The early model of reliability theory was primarily of series-parallel combinations of elements with exponential lifetimes. However, since the early 1960s there has been a rapid expansion around two new and important concepts. The first is the theory of coherent structures, which provides a general framework for analysing systems of unreliable elements. The second idea is to assume certain monotone shape properties of the lifetime distributions (increasing failure rate and rate average, increasing mean residual life, 'new better than used', etc.) to bound complex system performance measures, and determine optimal replacement policies (Barlow & Proschan 1965, 1975; Proschan 1976).

Current research topics are surveyed in Barlow & Proschan (1976). Of particular interest are a successful multivariate generalization of the failure rate (Marshall 1975), increased interest in Bayesian models (Tsokos 1977), and the use of fault trees to systematically develop failure modes in complex systems, such as nuclear reactors (Barlow, Fussell & Singpurwalla 1975).

14. Forecasting and Bayesian statistics

There are two developments in statistics which are influencing the methodology of operations research. In the empirical forecasting of time series, the ARIMA (auto-regressive integrated moving-average) models of Box & Jenkins (1970) provide an economical framework in which to identify reasonable underlying mechanisms and carry out the necessary computations. The theory also puts older exponential-smoothing heuristics on a firm basis.

The other development has been the so-called Bayesian revolution. Suppose we have some prior information about a random parameter $\tilde{\theta}$ (the inputs or control settings in a certain process, or the physical or economic conditions surrounding a certain experiment, or the skill of a human operator, etc.) which we can summarize in a prior density, $p(\theta)$; and suppose, for every possible value of $\tilde{\theta}$, we know the 'likelihood', $p(x|\tilde{\theta})$, the conditional density of observing a different random value, $\tilde{x}$, during some well-defined experiment. By the use of conditional expectation (Bayes's law), we find that, posterior-to-observing the sample value $\tilde{x} = x_0$, we can redefine our knowledge about θ to obtain the posterior-to-data density

$$p(\theta|x_0) = kp(x_0|\theta) . p(\theta), \tag{14.1}$$

where k is a constant to normalize $p(\theta|x_0)$. The current controversy in the statistical community seems to stem not from (14.1), but from whether a consulting statistician is permitted to have any personal beliefs about θ to include in the prior $p(\theta)$, or whether he must devise methods to let the data somehow 'speak for itself' (Savage *et al.* 1962; Barnett 1973).

This is hardly a crisis in operation research/systems analysis where the ability to draw on prior experience and analogous situations is permitted, nay, encouraged in estimation procedures. More importantly, the Bayesian approach reveals paradoxes in the classical sampling-theory school of statistics (Lindley 1972, 1975; Basu, 1975), and in spite of various attempts to reconcile the two approaches, such as the use of diffuse priors and 'empirical Bayes' techniques, it seems as if the basic sampling-theory ideas, suc h as point estimation, significance testing, and confidence intervals, must be reformulated. Houle (1973) gives about 2000 references in Bayesian statistics; many new references regularly appear in O.R. journals.

Motivated by estimation problems in insurance, the author has been interested in Bayesian prediction schemes, particularly the estimation of the mean value of a future observation posterior-to-data, namely

$$E\{\tilde{x}|x_0\} = \iint xp(x|\theta)\,p(\theta|x_0)\,\mathrm{d}x\,\mathrm{d}\theta, \tag{14.2}$$

which represents the 'experience-rated fair premium' in insurance terms. Actuaries noticed that, for many priors $p(\theta)$ and likelihoods $p(x|\theta)$, (14.2) was linear in the data x_0; this is true even for non-normal families, and the general conditions under which this is true are now known, and have been extended to the multi-dimensional case (Jewell 1974).

In more general forecasting and regression schemes, the Bayesian mean may not be linear in the data; however, one can easily find the best linear approximation through the use of least-squares theory. Thie field is referred to as 'credibility theory' in the actuarial literature (a survey is in Jewell 1976), and is closely allied with 'linear filter theory' in the communications field (Sage & Melsa 1971). Aitchison & Dunsmore (1975) analyse other Bayesian prediction schemes.

15. Industrial and management models

This section will briefly survey some of the recent trends in business models. The growth and influence of systems ideas over the last twenty years has been tremendous, and scientific management is now a fact of life in all kinds and sizes of industries.

Inventory control was already a mature field by the 1960s, following the development of the basic statistical models (Arrow *et al.* 1958; Scarf, Gilford & Shelly 1963). Since that time, the emphasis has been on making the models more realistic and extensive. Gross & Schrady (1976) is a recent summary; see also the forthcoming book by Silver & Peterson (1977). Muckstadt (1973) describes a large-scale application.

Turning to the production side, an important new trend has been the development of integrated systems for production planning, scheduling and inventory control (Hax & Golovin 1974; Hax & Mead 1975; Hax 1976*b*; Bitran & Hax 1976). The problems of logistics, including plant location and distribution, have also been the subject of recent intensive investigations (Eilon, Watson-Gandy & Christofides 1971; Francis & White 1974; Geisler 1975; Geoffrion 1975; Marlow 1976; see especially the survey by Hax 1976*a*). Multilevel analysis is described in Jennergren (1976).

Marketing, on the other hand, is an area which only recently has been quantified, apparently with success. Kotler (1971) provides a comprehensive survey; recent articles of interest are Little (1975) and Hauser & Urban (1976).

In the area of project management, the most important development of the 1960s was in scheduling, by using the critical-path methods developed for the Polaris missile program and the construction of the S.S. *France*. The basic models belong to the network class of linear programs, have very simple algorithms, and are now routinely used in all major construction projects (Thornley 1968; Moder & Phillips 1970; Lombaers 1969). Since that time, attention has been directed towards the resource-loading problem – an inherently difficult problem which is of the NP-complete class; Herroelen (1972) is a convenient survey. Shephard, Al-Ayat & Leachman (1976) is a different modelling approach, which uses dynamic production function theory. Another active area has been in the selection and budgeting of research and development projects (Gear, Lockett & Pearson 1971; Gear & Lockett 1973; Näslund & Sellstedt 1974; Baker & Freeland 1975). The problems of detailed manpower scheduling are also of continued interest (Bennett & Potts 1968; Arabeyre *et al.* 1969); Bodin (1972) gives a general model. Baker (1974) and Coffman (1975) are recent works on job/shop scheduling and sequencing.

Finally, the most explosive management science area in the past decade has been the field of investment and finance. Following the pioneering idea of Markowitz (1959) to select a portfolio of investments as a trade off between mean return and variance, many different extensions have been made in an attempt to improve investment performance (see Francis & Archer 1971; Sharpe 1971; Lorie & Brealey 1972). Part of the problem may be that the stock market is too efficient a process for a computer to make money, at least in the long run (Fama 1970; Granger & Morgenstern 1970). The optimal design of bond maturity schedules is, however, a more tractable problem (Bradley & Crane 1975). Money managers are also using linear programming and other methods to reduce or increase 'float' (Calman 1968; Orgler 1970; Orr 1971). A bibliography of 3600 works in the finance and investment area is in Brealey & Pyle (1973).

16. The influence of O.R. methodologies

Since other speakers at this meeting will be describing various applications of operational research/systems analysis, I would like to describe another process which has not been widely noted – the influence of O.R. methodologies upon other disciplines, particularly in research and teaching.

For example, we are so used to talking of the uses of mathematics, it is easy to overlook the stimulus that linear programming has given to the study of convex polytopes, solutions of inequalities, discrete mathematics, and graph theory. Linear algebra, including a brief introduction to linear programming, is now taught to all engineering freshmen and math majors at my university; simple graphical-solution linear programs even appear in high school 'new math' courses. New algorithms, based upon the complementary pivot theory of mathematical programming, give promise of providing practical calculation of fixed points – an achievement which will find wide application in both pure and applied mathematics (Karamaridan 1976; Saigal 1976).

Scarf (1973) has already applied these fixed-point algorithms to the computation of economic equilibria – a difficult problem which has heretofore eluded economists for even modest-sized problems. Duality theory, with its concepts of imputed values of resources and of 'pricing out' inefficient activities, has proved a fertile field for quantifying basic economic notions such as marginal costs. Large-scale economic planning and optimization is now possible on a scale undreamed of twenty years ago, thanks to modern linear programming codes. Production function theory has changed dramatically (Shaphard 1976). New terms, like trade-off, cost-benefit analysis, suboptimization, efficient frontier, and decentralized control, are universally used and understood.

In statistics, the various interesting problems posed by dynamic programming, decision analysis, Markov programming, etc., models have certainly stimulated research in statistical decision theory, gambling systems, martingales, potential theory, limit theorems in renewal theory, and so forth. The concept of monotone shape characteristics for distributions, introduced in reliability applications, has provided a new approach to bounding moments in random walks. And, the many possible variations in queueing models has provided a torrent of marginal contributions to the statistical journals, much to the concern of the editors (Pyke 1975).

I have already described the changes in almost every field of business administration; the same can be said about industrial engineering. The methodologies have also been adopted by electrical engineering, especially in control theory and in communications network design, as described earlier. Transportation engineering relies heavily upon queueing theory, network flows, dynamic programming, etc. (Gazis 1976). Critical path scheduling is taught routinely in construction engineering. Statistical models of wear are useful in metal behaviour studies. Dynamic programming is used for nuclear fuel management, and fault tree analysis to isolate nuclear reactor shutdown sequences. And so on.

A variety of new sister disciplines have also sprung up which use O.R. methodologies, as a glance at the new journals will reveal: urban planning, environmental engineering, energy analysis, health care systems, etc.

And especially in computer science it is possible to trace the influence of O.R. methodologies: from queueing theory for the design of computer systems, through graph theory and combinatorics for the design of efficient data structures and manipulation procedures, to the common concerns for developing, testing, and implementing efficient algorithms (Aho *et al.* 1975). It seems to me that there is a certain amount of tension just now between O.R. and this newest engineering science discipline, caused in part by the shift in popularity and research support, but also by the realization that problems of algorithmic efficiency have become too esoteric for the O.R. analyst, and require the attention of a different kind of specialist.

17. Crises within the profession

This tremendous activity and expansion in the field of operational research/systems analysis has, however, been achieved at the cost of considerable disorder within the profession, as the discussion sections of the journals and the conference round-table discussions reveal (see the references in Klein & Butkovitch 1976).

The first crisis is over the incredible proliferation in papers and specialty journals. O.R.S.A./T.I.M.S. conference dimensions are staggering, as are the numbers of regular meetings of numerous special-interest groups and new specialty societies. Kendall (1960) estimates that, in 1958, one would have to scan five journals to cover a third of the English-language contributions and about 18 journals to cover a half of the literature; my estimate of the current situation, based upon scanning our university libraries, is that about 15 and 50 journals, respectively, would be needed in 1976. Even traditional journals have split into several parts. One wonders what libraries can afford to stock them all, or how many people have 'xerox subscriptions'. Some will say that this proliferation is the direct result of the 'publish or perish' promotion criteria of American universities; others point to regulations requiring one to present a paper to secure travel support to a technical meeting. But it is clear that this communication explosion is affecting other sciences as well and there is no easy solution in sight.

With this proliferation has come increasingly narrow specialization in academic studies when students insist they want to major in mathematical programming or queueing theory, and the faculty advisors permit them to do so. Klein & Butkovitch (1976) suggests darkly that this is a natural phenomenon, since the O.R./M.S. academic discipline is an institutionalized system of exchange which sets up modular specializations in order to ensure its own survival; they see little hope for institutional change.

Another crisis, perhaps more pertinent to the U.S.A. than to the U.K., has been the apparent separation between theory and application. Practitioners regularly rage at the mathematical 'overkill' in the pages of the journals, and yearn for the good old days when a simple model could explicate an observed phenomenon; in rebuttal, researchers point to the trivial level of many of the applications papers ('How I ...'), and the universal lack of sponsorship by industry of meaningful research programmes.

There are continuing criticisms of the academic programme (Schrady 1976) which imply that O.R./M.S. training is not responsive to the needs of industry – being too technique-oriented, over-specialized, having little understanding of the total systems approach, unable to collect and organize data or write management reports, and so forth.

Putting all these tensions together, adding the success of new fields, such as computer science, and contemplating the rapid rate of adoption of O.R. methodologies by the applications fields (business; economics; transportation, environmental, and communications engineering, etc.), we are led to a larger malady, which might be called a crisis of confidence. We see this in the searching self-examination of many of the round-table discussions ('Is mathematical programming moribund?', 'Are we gambling on O.R./M.S. education?'), and in statements to the effect that operational research has promised too much, delivered too little, and should now be given a decent burial.

18. Future prospects

I prefer to take a somewhat more balanced view of these crises. For example, there is evidence that, in the U.S., the societies are moving to correct some of the earlier excesses. O.R.S.A. and T.I.M.S. have been growing more closely together, running simultaneous conferences, and sharing membership administration facilities. Their publication policies have also been co-ordinated and rationalized: a new journal, called *Mathematics of operations research*, has been established to attract important theoretical articles. *Interfaces* has been designated as the new joint medium for describing operational problems of implementing or using O.R./M.S.; the quality of articles is improving under the new editor, who insists that all articles be readable and that equations be relegated to the appendices. The parent journals, *Operations research* and *Management science*, are now free to concentrate on major articles of interest to all members of the profession.

Another interesting development has been the sponsorship of a prize competition for papers on successful applications of M.S., sponsored by the T.I.M.S. College on Practice (*Interfaces*, vol. 6, no. 1). The rules are strict: the entries must report a completed, practical application and must present results that have had a significant impact on the performance of the organization under study, as certified by management. Because practitioners do not normally publish such studies, the prize is set at a significant level ($6000 for the 1977 competition). The actual presentations and the written papers are extremely interesting, in my opinion, and provide a standard of professional practice previously unavailable.

There are no easy solutions to the publication explosion problem, although de-emphasis of published works as a university promotion criterion would certainly help, as would resolution of some of the legal and institutional problems surrounding the inexpensive duplication and distribution of papers. I believe in market mechanisms and the freedom to fail; journals which do not serve some useful purpose will soon vanish from the scene. Who is to question the utility of those which survive?

Academic programmes are easily criticized, but one must remember that there were few texts or courses before 1960, and new programmes had to be grafted onto a variety of different educational formats. It is true there has been a great deal of theoretical activity relative to the actual applications, but this is the characteristic of 'normal science' (Kuhn 1970), whose first priority is to structure the appropriate and potentially useful knowledge and explore its theoretical facets. 'Few people who are not actually practitioners of a mature science realize how much mop-up work of this sort a paradigm leaves to be done or quite how fascinating such work can prove in the execution' (Kuhn 1970, p. 24).

It is also true that our recent graduates, now staffing industry, government and other teaching faculties, have over-emphasized technique in place of application, and did not participate in the same school of hard knocks and simple models that reared our founders. But the same can be said of any profession. These young people are extremely bright, and, I believe, more adaptable to new demands by society than many of the tired pioneers. As far as teaching the systems approach is concerned, philosophy is fine (Churchman 1968), but what is needed are more excellent texts like White (1975), and good professional articles, developing ideas like those in Bishop (1972) and Liebman (1976). More consistent signals from research funding agencies would help useful academic development, as would more interest by industry in providing research problems and support, and helping educational programmes to make closer ties with reality.

O.R. academicians, on the other hand, must learn to let go of any proprietary feelings they have about the methodologies they helped to develop, and pay closer attention to the substantial issues facing their field of primary interest, be it business, government, or industrial engineering. The great strength of the profession has come from the ability to construct interesting models of real-world phenomena, and to use the solutions to resolve actual problems. The outlines of the applied methods are now clear for all to see, and, at some point, methodology becomes pure mathematics or statistics or computer science, the concern of other specialists. It is a mark of maturity that our methods are now influencing other fields, and that the availability of these new support skills frees us to return to the central issues of modelling and problem-solving.

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